1994: PART A

1) $f(x) = 3x^4 + x^3 - 21x^2$

a) $f'(x) = 12x^3 + 3x^2 - 42x$

$f'(2) = 12(2^3) + 3(2^2) - 42(2) = 12(8) + 3(4) - 84 = 24$

Equation of Tangent:

$y - (-28) = 24(x - 2)$

$y + 28 = 24x - 48$

$y = -24x - 76$

b) $f'(x) = 0$

$12x^3 + 3x^2 - 42x = 0$

$3x(4x^2 + x - 14) = 0$

$x(4x^2 + x - 14) = 0$

$x = 0 \quad , \quad 4x^2 + x - 14 = 0$

$x = -2, \ \frac{7}{4}$

$f'(x)$: $-$ -2 $+$ 0 $-$ $7/4$ $+$

\ _ / \ _ /

Rel. min @ $x = -2$, $x = \frac{7}{4}$

$f(-2) = 3(-2)^4 + (-2)^3 - 21(-2)^2 = -44$

$f(7/4) = 3(7/4)^4 + (7/4)^3 - 21(7/4)^2 = -35.457$

Since $f(x) \to \infty$ for $x \to \pm\infty$

Absolute min $= -44$

c) $f''(x) = 36x^2 + 6x - 42$

$f''(x) = 0$

$36x^2 + 6x - 42 = 0$

$6x^2 + x - 7 = 0$

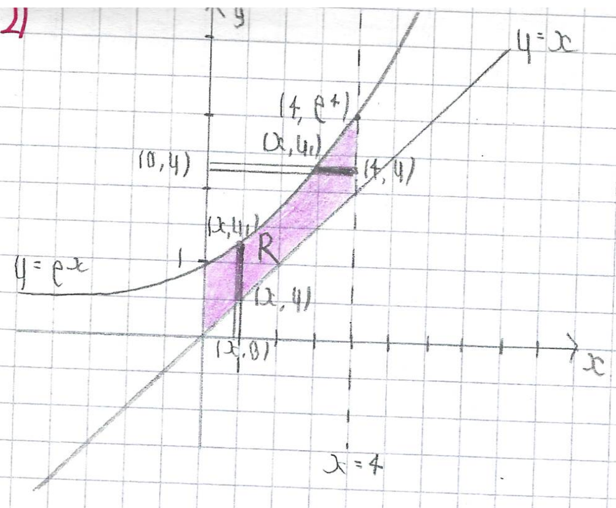
$x = 1, -\frac{7}{6}$

$f''(x)$: $+$ $-7/6$ $-$ 1 $+$

$\smile$ $\frown$ $\smile$

Pts. of inflection at $x = -\frac{7}{6}$, $x = 1$ since $f''(x)$ changes from conca up to concave down at $x = -\frac{7}{6}$ and concave down to concave up $x = 1$

2)



a) Area R $= \int_0^4 (e^x - x)\,dx$

$= \left[e^x - \frac{x^2}{2}\right]_0^4$

$= e^4 - 8 - e^0$

$= e^4 - 9$

b) $R(x) = y_1 - 0 = e^x$
$r(x) = y - 0 = x$

Volume $= \pi \int_0^4 ((e^x)^2 - x^2)\,dx$

$= \pi \int_0^4 (e^{2x} - x^2)\,dx$

$= \pi \left[\frac{e^{2x}}{2} - \frac{x^3}{3}\right]_0^4$

$= \pi \left(\frac{e^8}{2} - \frac{64}{3} - \frac{e^0}{2}\right)$

$= \pi \left(\frac{3e^8 - 128 - 3}{6}\right)$

$= \frac{\pi}{6}(3e^8 - 131)$

c) $R(x) = 4 - 0 = 4$ $\qquad y = e^x$ $\qquad$ * CANT DO * Shell
$r(x) = x - 0 = \ln y$ $\qquad x = \ln y$

Volume $= \pi \int$

3) $x^2 + xy + y^2 = 27$

a) $2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(x + 2y) = -2x - y$

$\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$

b) x intercept : $y = 0$
$x^2 = 27$
$x = \pm\sqrt{27} = \pm 3\sqrt{3}$

at $(3\sqrt{3}, 0)$:

$$\text{slope} = \frac{-6\sqrt{3} - 0}{3\sqrt{3} + 0} = -2$$

at $(-3\sqrt{3}, 0)$:

$$\text{slope} = \frac{6\sqrt{3} - 0}{-3\sqrt{3} + 0} = -2$$

$\therefore$ tangents are parallel

c) if vertical, slope is undefined

$x + 2y = 0$

$x = -2y$

$(-2y)^2 + (-2y)(y) + y^2 = 27$

$4y^2 - 2y^2 + y^2 = 27$

$3y^2 = 27$

$y^2 = 9$

$y = \pm 3$

$x = -2y$

$x = -2(3) = -6 \quad , \quad x = -2(-3) = 6$

$(-6, 3), (6, -3)$

Q4: PART B

4) $v(t) = t\ln t - t,\ t > 0,\ x(1) = 6$

a) $A(t) = v'(t)$

$A(t) = \ln t + t\left(\frac{1}{t}\right) - 1$

$= \ln t + 1 - 1$

$= \ln t$

b) $v(t) > 0$

$t\ln t - t > 0$

$t\ln t - t = 0$

$t(\ln t - 1) = 0$

$t \neq 0 \quad \ln t - 1 = 0$

$\ln t = 1$

$t = e^1$

$t = e$

v(t): sign diagram with points 0 and e: − between 0 and e, + after e

$t > e$

c) $v'(t) = 0$

$\ln t = 0$

$t = e^0$

$t = 1$

v'(t): sign diagram with points 0 and 1: − between 0 and 1, + after 1

Min since v'(t) changes sign from −ve to +ve

$v(1) = \ln 1 - 1 = 0 - 1 = -1$

Min velocity is −1

d) $x(t) = \int v(t)\,dt$

$x(t) = \int (t\ln t - t)\,dt \qquad$ let $u = \ln t \quad \frac{dv}{dt} = t$

$\frac{du}{dt} = \frac{1}{t} \quad v = \frac{t^2}{2}$

$x(t) = \ln t\left(\frac{t^2}{2}\right) - \int \frac{t^2}{2}\left(\frac{1}{t}\right)dt - \frac{t^2}{2}$

$= \frac{t^2\ln t}{2} - \int \frac{t}{2}\,dt - \frac{t^2}{2}$

$= \frac{t^2\ln t}{2} - \frac{t^2}{4} - \frac{t^2}{2} + c$

$x(1) = 6$

$6 = \frac{1}{2}\ln 1 - \frac{1}{4} - \frac{1}{2} + c$

$6 = -\frac{3}{4} + c$

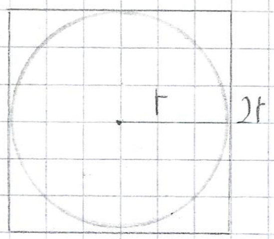
$24 = -3 + 4c$

$4c = 27$

$c = \frac{27}{4}$

$x(t) = \frac{t^2}{2}\ln t - \frac{3t^2}{4} + \frac{27}{4}$

5)



$\frac{dc}{dt} = 6$

a) we want $\frac{dp}{dt}$

$p = 8r$

$\frac{dp}{dt} = 8\frac{dr}{dt}$

$\frac{dp}{dt} = 8\left(\frac{3}{\pi}\right)$

$= \frac{24}{\pi}$ inches/sec

$c = 2\pi r$

$\frac{dc}{dt} = 2\pi\frac{dr}{dt}$

$6 = 2\pi\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{3}{\pi}$

b) when $A = 25\pi$

$\pi r^2 = 25\pi$

$r^2 = 25$

$r = 5$

Area circle $= \pi r^2$

Area square $= 4r^2$

Area enclosed $= 4r^2 - \pi r^2$

$A_2 = 4r^2 - \pi r^2$

$\frac{dA_2}{dt} = 8r\frac{dr}{dt} - 2\pi r\frac{dr}{dt}$

$= 8(5)\left(\frac{3}{\pi}\right) - 2\pi(5)\left(\frac{3}{\pi}\right)$

$= \frac{120}{\pi} - 30$

6) $F(x) = \int_0^x \sin(t^2)dt$ for $0 \le x \le 3$

a) $F(1) = \int_0^1 \sin(t^2)dt$

$= \frac{1-0}{2(4)}\left[F(0) + 2(F(0.25) + F(0.5) + F(0.75)) + F(1)\right]$

$= \frac{1}{8}\left[0 + 2(\sin(1/16) + \sin(1/4) + \sin(9/16)) + \sin(1)\right]$

$= 0.314$ to 3.d.p.

b) increasing when $F'(x) > 0$

$F'(x) = \frac{d}{dx}\sin(t^2) = \sin(x^2)$

$0 \le x^2 \le 9$

$0 \le x \le 3$

$\sin(x^2) > 0$

$x^2 = 0, \pi, 2\pi, \ldots = n\pi$

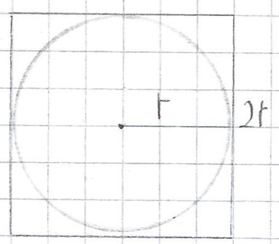
$x = 0, \sqrt{\pi}, \sqrt{2\pi}$

$F'(x)$: $+$ $-$ $+$

0 $\sqrt{\pi}$ $\sqrt{2\pi}$ 3

increasing on $[0, \sqrt{\pi}), (\sqrt{2\pi}, 3]$

5)



$\frac{dc}{dt} = 6$

a) we want $\frac{dp}{dt}$

$p = 8r$

$\frac{dp}{dt} = 8\frac{dr}{dt}$

$\frac{dp}{dt} = 8\left(\frac{3}{\pi}\right)$

$= \frac{24}{\pi}$ inches/sec

$c = 2\pi r$

$\frac{dc}{dt} = 2\pi \frac{dr}{dt}$

$6 = 2\pi \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{3}{\pi}$

b) when $A = 25\pi$

$\pi r^2 = 25\pi$

$r^2 = 25$

$r = 5$

Area circle $= \pi r^2$

Area square $= 4r^2$

Area enclosed $= 4r^2 - \pi r^2$

$A_2 = 4r^2 - \pi r^2$

$\frac{dA_2}{dt} = 8r\frac{dr}{dt} - 2\pi r\frac{dr}{dt}$

$= 8(5)\left(\frac{3}{\pi}\right) - 2\pi(5)\left(\frac{3}{\pi}\right)$

$= \frac{120}{\pi} - 30$

6) $F(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$

a) $F(1) = \int_0^1 \sin(t^2)\,dt$

$= \frac{1-0}{2(4)}\left[F(0) + 2(F(0.25) + F(0.5) + F(0.75)) + F(1)\right]$

$= \frac{1}{8}\left[0 + 2(\sin(1/16) + \sin(1/4) + \sin(9/16)) + \sin(1)\right]$

$= 0.314$ to 3.d.p.

b) increasing when $F'(x) > 0$

$F'(x) = \frac{d}{dx}\sin(t^2) = \sin(x^2)$

$0 \le x^2 \le 9$

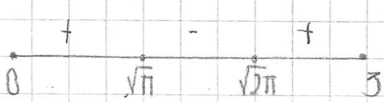
$0 \le x \le 3$

$\sin(x^2) > 0$

$x^2 = 0, \pi, 2\pi, \ldots = n\pi$

$x = 0, \sqrt{\pi}, \sqrt{2\pi}$

$F'(x)$:



increasing on $[0, \sqrt{\pi}), (\sqrt{2\pi}, 3]$

Q4: PART B

4) $v(t) = t\ln t - t,\ t > 0,\ x(1) = 6$

a) $A(t) = v'(t)$

$A(t) = \ln t + t\left(\frac{1}{t}\right) - 1$

$= \ln t + 1 - 1$

$= \ln t$

b) $v(t) > 0$

$t\ln t - t > 0$

$t\ln t - t = 0$

$t(\ln t - 1) = 0$

$t \neq 0 \qquad \ln t - 1 = 0$

$\ln t = 1$

$t = e^1$

$t = e$

$v(t)$: 0 $-$ e $+$

$t > e$

c) $v'(t) = 0$

$\ln t = 0$

$t = e^0$

$t = 1$

$v'(t)$: 0 $-$ 1 $+$

Min since $v'(t)$ changes sign from $-$ve to $+$ve

$v(1) = 1\ln 1 - 1 = 0 - 1 = -1$

Min velocity is -1

d) $x(t) = \int v(t)\,dt$

$x(t) = \int (t\ln t - t)\,dt \qquad$ let $u = \ln t \qquad \frac{dv}{dt} = t$

$\frac{du}{dt} = \frac{1}{t} \qquad v = \frac{t^2}{2}$

$x(t) = \ln t\left(\frac{t^2}{2}\right) - \int \frac{t^2}{2}\left(\frac{1}{t}\right)dt - \frac{t^2}{2}$

$= \frac{t^2}{2}\ln t - \int \frac{t}{2}\,dt - \frac{t^2}{2}$

$= \frac{t^2}{2}\ln t - \frac{t^2}{4} - \frac{t^2}{2} + C$

$x(1) = 6$

$6 = \frac{1}{2}\ln 1 - \frac{1}{4} - \frac{1}{2} + C$

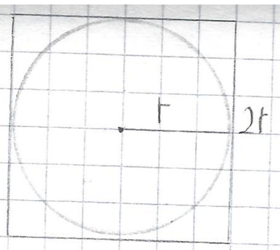
$6 = -\frac{3}{4} + C$

$24 = -3 + 4C$

$4C = 27$

$C = \frac{27}{4}$

$x(t) = \frac{t^2}{2}\ln t - \frac{3t^2}{4} + \frac{27}{4}$



$\frac{dc}{dt} = 6$

a) we want $\frac{dp}{dt}$

$p = 8r$

$\frac{dp}{dt} = 8\frac{dr}{dt}$

$\frac{dp}{dt} = 8\left(\frac{3}{\pi}\right)$

$= \frac{24}{\pi}$ inches/sec

$C = 2\pi r$

$\frac{dc}{dt} = 2\pi\frac{dr}{dt}$

$6 = 2\pi\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{3}{\pi}$

b) when $A = 25\pi$

$\pi r^2 = 25\pi$

$r^2 = 25$

$r = 5$

Area circle $= \pi r^2$

Area square $= 4r^2$

Area enclosed $= 4r^2 - \pi r^2$

$A_2 = 4r^2 - \pi r^2$

$\frac{dA_2}{dt} = 8r\frac{dr}{dt} - 2\pi r\frac{dr}{dt}$

$= 8(5)\left(\frac{3}{\pi}\right) - 2\pi(5)\left(\frac{3}{\pi}\right)$

$= \frac{120}{\pi} - 30$

c) $F(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$

a) $F(1) = \int_0^1 \sin(t^2)\,dt$

$= \frac{1-0}{2(4)}\left[F(0) + 2(F(0.25) + F(0.5) + F(0.75)) + F(1)\right]$

$= \frac{1}{8}\left[0 + 2(\sin(1/16) + \sin(1/4) + \sin(9/16)) + \sin(1)\right]$

$= 0.314$ to 3.d.p.

b) increasing when $F'(x) > 0$

$F'(x) = \frac{d}{dx}\sin(t^2) = \sin(x^2)$

$0 \le x^2 \le 9$

$0 \le x \le 3$

$\sin(x^2) > 0$

$x^2 = 0, \pi, 2\pi, \ldots = n\pi$

$x = 0, \sqrt{\pi}, \sqrt{2\pi}$

$F'(x)$: + − +

0 $\sqrt{\pi}$ $\sqrt{2\pi}$ 3

increasing on $[0, \sqrt{\pi}), (\sqrt{2\pi}, 3]$

$$\text{Av} = \frac{F(3)-F(1)}{3-1} = K$$

$$F(3) = \int_0^3 \sin(t^2)\,dt \qquad F(1) = \int_0^1 \sin(t^2)\,dt$$

$$F(3)-F(1) = \int_0^3 \sin(t^2)\,dt - \int_0^1 \sin(t^2)\,dt$$

$$= \int_1^3 \sin(t^2)\,dt$$

$$= 2K$$

OR:

$$\frac{\int_1^3 \sin(x^2)\,dx}{3-1} = \frac{\int_1^3 F'(x)\,dx}{2} = \frac{[f(x)]_1^3}{2} = \int_1^3 \sin(x^2)\,dx = 2K$$